ORIENTATION AND SYMMETRIES OF ALEXANDROV SPACES WITH APPLICATIONS IN POSITIVE CURVATURE

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ABSTRACT. We develop new tools for use in Alexandrov geometry and apply them to the problem of classifying positively curved Alexandrov spaces with maximal symmetry rank: the first being a theory of ramified orientable double covers and the second being a particularly useful version of the Slice Theorem for actions of compact Lie groups.

These techniques are applied to compact, positively curved Alexandrov spaces to provide a description of fixed-point homogeneous spaces as the quotient of a join, and to show that the maximal symmetry rank of a compact, positively curved Alexandrov space is the same as for the Riemannian case. The spaces of maximal symmetry rank are shown to be quotients of spheres by rank 0 or 1 subgroups of the centralizer of the maximal torus in the orthogonal group.

INTRODUCTION

Certain techniques are so well-established and of such utility when studying the geometry of Riemannian manifolds that when working in the more general area of Alexandrov geometry one reaches reflexively for them, only to find that they are not within reach. Of course, the increased generality of the subject necessitates weaker results. The existence of convexity and injectivity radii, extendibility of geodesics and isotopies via vector fields are all foregone, but in return we have a richer class of spaces with which to work.

Other tools, however, we may hope to retain, though often in a modified form. In this paper we present two important examples: the theory of orientable double covers and the Slice Theorem, with the required modifications.

It is well known that every Riemannian manifold M has an orientable double cover $\tilde{M}$, and that there is a free orientation-reversing isometric involution $i : \tilde{M} \to \tilde{M}$ such that M is isometric to $\tilde{M}/i$. One of the useful

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aspects of the class of Alexandrov spaces is that it is closed under taking quotients by isometric group actions even when those actions are not free. Therefore it is natural to allow the involution i to fix points. We obtain the following result.

Theorem A. *Let X be an Alexandrov space of dimension n and $\mathrm{curv} \geq k$ which is non-orientable. Then there is an orientable Alexandrov space $\tilde{X}_{Ram}$ with the same dimension and lower curvature bound, and with an isometric involution i such that $\tilde{X}_{Ram}/i$ and X are isometric. $\tilde{X}_{Ram}$ is a ramified orientable double cover of X, and the ramification locus is the union of those strata in X having non-orientable normal cones.*

When studying the action of a compact Lie group on a topological space, the Slice Theorem is a crucial component of the theory. It is clear that the theorem is true in Alexandrov geometry, simply because Alexandrov spaces are completely regular. However, in Riemannian geometry we can go further, identifying the slice with the normal space to the orbit. We show that an analogous identification is possible in Alexandrov geometry.

Theorem B. *Let a compact Lie group G act isometrically on an Alexandrov space X. Then for all $p \in X$, there is some $r_0 > 0$ such that for all $r < r_0$ there is an equivariant homeomorphism $\Phi : G \times_{G_p} K\nu_p \to B_r(G(p))$ where ν_p is the space of normal directions to the orbit $G(p)$.*

Recall that the group of isometries of an Alexandrov space is a Lie group [7], just as for Riemannian manifolds [25]. The project of classifying positively curved Riemannian manifolds with "large" isometry groups (where the largeness of the group action may be interpreted in a variety of ways) can therefore reasonably be extended to positively curved Alexandrov spaces. It is of great interest to see which positively curved singular spaces can arise in the presence of symmetries. Where we find few additional spaces, we may conclude that the restrictions are due to the nature of curvature and symmetry, while where there are many additional examples we may conclude that the restriction has more to do with the nature of Riemannian manifolds.

One measurement for the size of a transformation group $G \times X \to X$ is the dimension of its orbit space X/G, also called the *cohomogeneity* of the action. In particular, the spaces of cohomogeneity 0 are the homogeneous spaces.

Berestovskiĭ has shown (modulo a mild condition verified in [23]) that finite-dimensional homogeneous spaces with a lower curvature bound are Riemannian manifolds [2]. In [10] it was shown that the maximum dimension for the isometry group of an Alexandrov space is the same as in the Riemannian case and that when this dimension is achieved the Alexandrov space is homogeneous.

In contrast to this, we have Alexandrov spaces of cohomogeneity one, which were studied in [12]. It was shown that non-manifold Alexandrov spaces of cohomogeneity one exist in all dimensions greater than or equal to 3. In particular, the suspension of any positively curved homogeneous space is a positively curved space of cohomogeneity one.

The cohomogeneity is clearly constrained by the dimension of the fixed point set X^G of G in X (where we understand the dimension of the empty set to be -1). In fact, $\dim(X/G) \geq \dim(X^G)+1$ for any non-trivial action. In light of this, the *fixed-point cohomogeneity* of an action, denoted by $\mathrm{cohomfix}(X;G)$, is defined by

$$\mathrm{cohomfix}(X;G) = \dim(X/G) - \dim(X^G) - 1 \geq 0.$$

A space with fixed-point cohomogeneity 0 is called *fixed-point homogeneous*.

We first recall the classification result for fixed-point homogeneous Riemannian manifolds of positive sectional curvature [16].

Theorem 0.1. *Let a compact Lie group G act isometrically and fixed-point homogeneously on M^n, a closed, simply-connected positively curved Riemannian manifold. Then M^n is diffeomorphic to one of S^n, $\mathbb{C}P^k$, $\mathbb{H}P^m$ or CaP^2, where $2k = 4m = n$.*

This result is obtained via a structure theorem which allows us to decompose the manifold as the union of two disk bundles. The natural generalization of this structure theorem to Alexandrov spaces would involve replacing the disk bundles with more general cone bundles. However, the rich variety of spaces of directions in Alexandrov geometry means that the structure of the orbit space can be more complicated than in the Riemannian setting, and so the result fails. We provide here an alternative description which does generalize to Alexandrov spaces.

Theorem C. *Let a compact Lie group G act isometrically and fixed-point homogeneously on X^n, a compact n-dimensional Alexandrov space of positive curvature and assume that $X^G \neq \emptyset$. If $H \subset G$ is the principal isotropy and F is the component of X^G with maximal dimension then the following hold:*

(i) *There is a unique orbit $G(p) \cong G/G_p$ at maximal distance from F (the "soul" orbit).*
(ii) *All principal G_p-orbits in ν, the normal space of directions to $G(p)$ at p, are homeomorphic to G_p/H. Moreover F is homeomorphic to ν/G_p.*
(iii) *The space X is G-equivariantly homeomorphic to*

$$(\nu * G)/G_p,$$

where G_p acts on ν with the isotropy action at p and on G by its left action. The G-action is induced by the action on $\nu * G$ given by the join of the trivial action and the left action.

(iv) *The principal orbits in $X \setminus (F \cup G(p))$ are homeomorphic to $\nu(F) \cong G/H$, where $\nu(F)$ is the positively curved space of normal directions to F.*

We see from this that fixed-point homogeneous spaces are plentiful among the positively curved Alexandrov spaces. For every positively curved space ν, its join to a positively curved homogeneous G-space yields a fixed-point homogeneous G-space. It appears from this result that fixed-point homogeneity is only a highly restrictive measure of symmetry in the context of Riemannian manifolds.

Another possible measure of symmetry is the *symmetry rank* of the space, where

$$\text{symrank}(X) = \text{rk}(\mathrm{Isom}(X)).$$

Closed Riemannian manifolds with positive curvature and maximal symmetry rank were classified in [15].

Maximal Symmetry Rank Theorem. *Let M be an n-dimensional, closed, connected Riemannian manifold with positive sectional curvature. Then*

(1) *symrank$(M) \leq \lfloor \frac{n+1}{2} \rfloor$.*

(2) *Moreover, equality holds in (1) only if M is diffeomorphic to a sphere, a real or complex projective space or a lens space.*

Observe that all such manifolds may be written as quotients of spheres by freely acting subgroups of the orthogonal group which commute with the maximal torus. Indeed, it is clear from the proof that the group actions are always induced by the maximal torus in the orthogonal group (cf. also [8], or [24], noting that the latter contains a more general result applicable to the spherical space-forms).

The list of maximal symmetry rank spaces is short because there are so few subgroups which satisfy this condition. In $O(2n)$ the maximal torus is its own centralizer, and its freely acting subgroups are finite cyclic or the diagonal circle. In $O(2n + 1)$ no subgroup of the maximal torus can act freely. However, the maximal torus does commute with the antipodal map, and so it follows that the real projective spaces have maximal symmetry rank.

Just as we allowed the isometric involution to act with fixed points in Theorem A, we can see immediately that a maximal symmetry rank space will arise whenever we take the quotient of a sphere by an appropriately sized subgroup of the centralizer of the maximal torus in the orthogonal group. In fact, using inductive methods relying on Theorem C, we show that

these spaces are the only positively curved Alexandrov spaces of maximal symmetry rank.

Theorem D. *Let X be an n-dimensional, compact, Alexandrov space with $\operatorname{curv} \geq 1$ admitting an isometric, (almost) effective T^k-action. Then $k \leq \lfloor \frac{n+1}{2} \rfloor$ and in the case of equality either*

(1) *X is a spherical orbifold, homeomorphic to S^n/G, where G is a finite subgroup of the centralizer of the maximal torus in $O(n+1)$ or*
(2) *only in the case that n is even, $X \cong S^{n+1}/G$, where G is a rank one subgroup of the maximal torus in $O(n+2)$.*

In both cases the action on X is equivalent to the action induced by the maximal torus on the G-quotient of the corresponding sphere.

Non-orientable spaces of maximal symmetry rank can only arise when G contains orientation-reversing elements, which only occurs in even dimensions and therefore all odd-dimensional maximal symmetry rank spaces are orientable. Note that this result is sharp: there are locally non-orientable odd-dimensional Alexandrov spaces admitting actions of almost maximal symmetry rank, that is, of rank $\lfloor \frac{n-1}{2} \rfloor$, such as, for example, in dimension 3, $\Sigma(\mathbb{R}P^2)$. Further, we can see that in dimensions ≤ 3 Alexandrov spaces of maximal symmetry rank must be topological manifolds, whereas in dimension 4, there are numerous examples of Alexandrov spaces of maximal symmetry rank that are not manifolds. However, since these spaces are all ultimately spherical in origin, we can say that the combination of positive curvature with maximal symmetry rank is restrictive in its own right, and not only in the Riemannian setting.

The paper is organized as follows. In Section 1 we will recall some general facts about Alexandrov spaces. In Section 2 we develop the theory of ramified orientable double covers, proving Theorem A. In Section 3 we will consider isometric group actions on Alexandrov spaces and prove Theorem B, as well as generalizations of other well-known results from the Riemannian case. In Section 4 we prove Theorem C, which is then applied in Section 5 to prove Theorem D.

Finally, we note that in a forthcoming paper [18] we will prove the following result for positively curved spaces having almost maximal symmetry rank in low dimensions. These spaces are also very restrictive, and spherical in origin. We observe that for the more restrictive case of positively curved, 4-dimensional, simply-connected topologically regular Alexandrov spaces, it has been shown that the only such spaces admitting an isometric circle action are S^4 and $\mathbb{C}P^2$ [9].

Theorem 0.2. *Let T^1 act isometrically and effectively on X^4, or T^2 on X^5, where X is a compact, positively curved, orientable Alexandrov spaces. Then up to homeomorphism*

(1) *X^4 is S^5/G where G is a rank one subgroup of the centralizer of any $T^2 \subset SO(6)$, or any orientable suspension (which corresponds to G being a finite subgroup of the centralizer of a $T^1 \subset SO(5)$; and*
(2) *X^5 is S^5/G where G is a finite subgroup of the centralizer of any T^2 in $SO(6)$, or S^6/G where G is a rank one subgroup of the centralizer of any $T^3 \subset SO(7)$ or S^7/G where G is a rank two subgroup of any $T^4 \subset SO(8)$.*

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1. Preliminaries

In this section we will first fix notation and then recall basic definitions and theorems about Alexandrov spaces.

We will denote an Alexandrov space by X, and will always assume it is complete and finite-dimensional. Given an isometric (left) action $G \times X \to X$ of a Lie group G, and a point $p \in X$, we let $G(p) = \{ gp : g \in G \}$ be the *orbit* of p under the action of G. The *isotropy group* of p is the subgroup $G_p = \{ g \in G : gp = p \}$. Recall that $G(p) \cong G/G_p$. We will denote the orbit space of this action by $\bar{X} = X/G$. Similarly, the image of a point $p \in X$ under the orbit projection map $\pi : X \to \bar{X}$ will be denoted by $\bar{p} \in \bar{X}$. We will assume throughout that G is compact and in Section 5 that its action is either *effective* or *almost effective*, i.e., that $\bigcap_{p\in X} G_p$ is respectively either trivial or a finite subgroup of G.

We will always consider the empty set to have dimension -1. Homeomorphisms and isomorphisms will be represented by $\cong$, while isometries will be represented by $=$. We will use T^1 to refer to the circle as a Lie group, and S^1 to refer to it as a topological space without any group structure.

1.1. **Alexandrov geometry.** A finite-dimensional *Alexandrov space* is a locally complete, locally compact, connected (except in dimension 0, where a two-point space is admitted by convention) length space, with a lower curvature bound in the triangle-comparison sense. Like most authors, we will assume that the space is complete. For non-complete spaces, we will

follow [34] in using the term *Alexandrov domain*. Every point in an Alexandrov domain has a closed neighborhood which is an Alexandrov space [32]. There are a number of introductions to Alexandrov spaces to which the reader may refer for basic information (cf. [5, 6, 27, 38, 39]).

A more analytic formulation of the curvature condition, using the concavity of distance functions, was introduced in [33]. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is λ-*concave* if it satisfies the differential inequality $f'' \leq \lambda$, in the barrier sense. A function $f : X \to \mathbb{R}$ on a length space X is λ-concave if its restriction to every geodesic is λ-concave. We use the term *semi-concave* to describe functions which are locally λ-concave, where λ need not have a uniform upper bound. Semi-concave functions on Alexandrov spaces have a well-defined gradient, and in particular, this gives rise to a gradient flow.

We say that a complete length space X is an Alexandrov space with $\operatorname{curv}(X) \geq k$ if, for any point p, a modification of dist$(p, \cdot)$ satisfies a certain concavity condition. In particular, if we let $f = \rho_k \circ$ dist$(p, \cdot)$, where

$$\rho_k(x) = \begin{cases} 1/k(1 - \cos(x\sqrt{k})), & \text{if } k > 0 \\ x^2/2, & \text{if } k = 0 \\ 1/k(1 - \cosh(x\sqrt{-k})), & \text{if } k < 0, \end{cases} \tag{1}$$

then f must be $(1 - kf)$-concave. Note that in a space form of constant curvature k equality holds, that is, $f'' = 1 - kf$.

The *space of directions* of an Alexandrov space X^n of dimension n at a point p is, by definition, the completion of the space of geodesic directions at p and is denoted by $\Sigma_p X$ or, if there is no confusion, Σ_p. For any subset Y of X^n, we denote the space of directions tangent to Y at $p \in Y$ by $\Sigma_p Y$. The space of directions of X^n is a compact Alexandrov space of dimension $n - 1$ with $\operatorname{curv} \geq 1$. We recall here a particularly useful result for such spaces (cf. [14, 13]).

Lemma 1.1 (Join Lemma)**.** *Let X be an n-dimensional Alexandrov space with $\operatorname{curv} \geq 1$. If X contains an isometric copy of the unit round sphere S_1^m, then X is isometric to the spherical join $S_1^m * \nu$, where ν is an isometrically embedded $(n-m-1)$-dimensional Alexandrov space with $\operatorname{curv} \geq 1$ which we will refer to as the* normal space *to S_1^m .*

In order to understand Alexandrov spaces, a grasp of their local structure is required. Perelman showed in [27] that Alexandrov spaces are spaces with multiple conic singularities (MCS spaces) in the sense of Siebenmann [40].

Definition 1.2. X is a *space with multiple conic singularities* or *MCS space* of dimension n if and only if every point $p \in X$ has a neighborhood which

is pointed homeomorphic to an open cone on a compact MCS space of dimension $n-1$, where the unique MCS space of dimension -1 is the empty set.

For Alexandrov spaces, the cone here can be taken to be the cone on the space of directions. This surprisingly difficult result is obtained in [27] (cf. also [20]).

An important difference between Riemannian manifolds and Alexandrov spaces is the existence of singularities. We refer to a point $p \in X$ as *regular* if Σ_p is isometric to the unit round sphere and as *singular* otherwise. We make the further distinction that a point is *topologically singular* if its space of directions is not homeomorphic to a sphere. The set of regular points of an Alexandrov space is dense and convex, while the singular points "may be arranged chaotically" [32].

By restricting our attention to certain kinds of singularities, we can stratify an Alexandrov space into manifolds in two different ways: the first stratification is purely topological and the second is by extremal sets.

The canonical stratification into manifolds of an MCS space is given as follows: a point $p \in X$ belongs to the l-dimensional stratum $X^{(l)}$ if p has a conic neighborhood homeomorphic to $\mathbb{R}^l \times K$, where l has been chosen to be maximal and K is a cone on a compact MCS space. We will describe K as the normal cone to the stratum $X^{(l)}$. For more information on this topological stratification of Alexandrov spaces, see [28]. Using this stratification we can see that the codimension of the set of topologically singular points (other than boundary points) is at least 3.

The more refined stratification by *extremal sets*, which takes into account metric information, is given in [32]. A non-empty, proper extremal set comprises points with spaces of directions which differ significantly from the unit round sphere. They can be defined as the sets which are "ideals" of the gradient flow of $\mathrm{dist}(p,\cdot)$ for every point p. Examples of extremal sets are isolated points with spaces of directions of diameter $\leq \pi/2$, the boundary of an Alexandrov space and, in a trivial sense, the entire Alexandrov space. The restriction of this stratification to the boundary partitions the boundary into *faces*. We refer the reader to [35] for definitions and important results.

We introduce here a notion of regular points in extremal sets, which will be useful later. A point $p \in E^k \subset X^n$ will be called E*-regular* when $\Sigma_p E = S^k_1$, that is, the space of directions tangent to E at the point p is isometric to a unit round sphere.

Proposition 1.3. *Let X be an Alexandrov space with boundary and $E = \partial X$. Then E contains a dense set of E-regular points.*

Proof. Consider $D(X) = X \cup_{\partial X} X$, the double of X, formed by identifying two copies of X along their common boundaries, ∂X. Since $D(X)$ is also

an Alexandrov space [27], it has a dense set of regular points, and, since that set is convex, $\partial X \subset D(X)$ also has a dense set of regular points. Then in X these points will be E-regular. □

When an extremal set is given its intrinsic path metric, the shortest paths in the extremal set exhibit characteristics similar to geodesics. We can generalize the notion of geodesic to include these "quasigeodesics" as follows.

Definition 1.4. A curve γ in X, an Alexandrov space with $\operatorname{curv} \geq k$, is a *quasigeodesic* if and only if it is parametrized by arc-length and $f(t) = \rho_k \circ \operatorname{dist}(p, \gamma(t))$ is $(1 - kf)$-concave, where ρ_k is the function defined in Equation 1.

The natural generalization of totally geodesic submanifolds from Riemannian geometry is the totally quasigeodesic subset. We say that a closed subset $Y \subset X$ is totally quasigeodesic if a shortest path in Y between points is a quasigeodesic in the ambient space X. See [35] for the formal definition. Extremal sets are the most important example of totally quasigeodesic subsets.

1.2. **Transformation Groups and Alexandrov Spaces.** We will now concentrate our attention on isometric group actions on Alexandrov spaces.

The following important proposition from [12] uses the Join Lemma 1.1 to describe the tangent and normal spaces to an orbit of an isometric group action.

Proposition 1.5. *Let X be an Alexandrov space admitting an isometric action of a compact Lie group G and fix $p \in X$ with dim$(G/G_p) > 0$. If $S_p \subset \Sigma_p$ is the unit tangent space to the orbit $G(p) \cong G/G_p$, then the following hold.*

(1) *S_p is isometric to the unit round sphere.*
(2) *The set $\nu(S_p)$ is a compact, totally geodesic Alexandrov subspace of $\Sigma_p X$ with $\operatorname{curv} \geq 1$, and the space of directions $\Sigma_p X$ is isometric to the join $S_p * \nu(S_p)$ with the standard join metric.*
(3) *Either $\nu(S_p)$ is connected or it contains exactly two points at distance π.*

It was shown in [10] that if G acts effectively on X then the induced isometric action of G_p on Σ_p must also be effective. Therefore G_p will act isometrically and effectively on the space of normal directions $\nu(S_p)$, which we will simply write as ν_p from now on.

1.3. **The Soul Theorem.** The following important result for non-negatively curved Alexandrov spaces will be used throughout the text [27].

Soul Theorem 1.6. *Let X be a compact Alexandrov space of* $\mathrm{curv} \geq 0$ *and suppose that $\partial X \neq \emptyset$. Then there exists a totally convex, compact subset $S \subset X$, called* the soul *of X, with $\partial S = \emptyset$, which is a strong deformation retract of X. If* $\mathrm{curv}(X) > 0$*, then the soul is a point, that is, $S = \{s\}$.*

The proof relies on the concavity of dist$(\partial X, \cdot)$. The gradient flow of the distance function on $X \setminus \partial X$ is 1-Lipschitz, and so can be extended to all of X. This flow plays the role of the Sharafutdinov retraction. Note that instead of using the distance from ∂X we may use the distance from any union of boundary faces (see [43]). When $\mathrm{curv}(X) > 0$ the boundary has only one component and is homeomorphic to Σ_s.

In the special case where X is the quotient space of an isometric group action on an Alexandrov space Y with $\mathrm{curv} > 0$, that is, $\pi : Y \to X = Y/G$ and $\partial X \neq \emptyset$, we see that $\partial X \cong \Sigma_s \cong \nu(G(p))/G_p$, where $p \in \pi^{-1}(s)$. Because the gradient flow preserves extremal subsets, it is clear that the orbit space has a somewhat conical structure. The non-principal orbit types are either s, contained within ∂X, or stretch from ∂X to s.

1.4. **Orientability of Alexandrov spaces.** As Petrunin has pointed out in [34], Alexandrov spaces are unlike manifolds in that they can have arbitrarily small neighborhoods which do not admit an orientation. In particular, if $p \in X$ has a non-orientable space of directions Σ_p, then no neighborhood of p is orientable. We call such a space *locally non-orientable*. We will use Alexander-Spanier cohomology to study orientability, as it has certain advantages in the context of Alexandrov spaces (cf. [14]), and coincides with singular cohomology.

It is easy to see by excision that $H^n(X, X \setminus \{p\}; \mathbb{Z}) \cong H^{n-1}(\Sigma_p; \mathbb{Z})$. If $H^{n-1}(\Sigma_p; \mathbb{Z}) \cong \mathbb{Z}$, then a choice of generator of $H^{n-1}(\Sigma_p; \mathbb{Z})$ is a *local orientation* at p. We will say that X is *locally orientable* if a local orientation can be chosen at each point $p \in X$.

We will define orientability of a compact Alexandrov space without boundary, X, in terms of the existence of a fundamental class. That is, as in the manifold case, X is orientable if for every $x \in X$, $H^{n-1}(X, X \setminus \{p\}; \mathbb{Z}) \to H^n(X; \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism. For non-compact Alexandrov spaces without boundary we define orientability using cohomology with compact supports. If the space has boundary, we will use relative cohomology. Henceforth we will use integer coefficients.

1.5. **Classical Theorems for Positive Curvature.** For a Riemannian manifold of positive sectional curvature there are two important theorems that characterize its topology. They are the Bonnet-Myers theorem, which tells us that if the curvature is bounded away from zero the manifold is compact and the fundamental group is finite, and Synge's theorem, which tells

us that in even dimensions an orientable manifold of positive curvature is simply connected and in odd dimensions a manifold of positive curvature is orientable.

Petrunin [34] proved an analogue of Synge's theorem for locally orientable Alexandrov spaces, which we recall here without the hypothesis of local orientability in even dimensions, a simple improvement which follows directly from the results in Section 2.

Generalized Synge's Theorem 1.7. *Let X^n be an Alexandrov space with* $\operatorname{curv} \geq 1$.

(1) *If X is even-dimensional and is either orientable or locally non-orientable then X is simply connected, otherwise it has fundamental group $\mathbb{Z}_2$.*
(2) *If X is odd-dimensional and locally orientable then X is orientable.*

The analogue of the Bonnet-Myers theorem for general Alexandrov spaces is well known but could not be located by the authors elsewhere in the literature. For completeness, a proof is presented here.

Generalized Bonnet-Myers Theorem 1.8. *Let X be an Alexandrov space of $\operatorname{curv} \geq k > 0$. Then X is compact and has finite fundamental group.*

Proof. Since an Alexandrov space of $\operatorname{curv} \geq k > 0$ has diameter bounded above by $\pi/\sqrt{k}$ [6], it follows from local compactness that X must be compact. Since Alexandrov spaces are MCS spaces, they have universal covers (cf. [3]), and it is clear that the metric on X induces a metric on the universal cover with the same curvature bound. The proof now proceeds just as in the Riemannian case. □

Observation. Unlike in the manifold case, simple connectivity does not imply orientability for Alexandrov spaces in general. The universal cover of a non-orientable Alexandrov space can therefore be non-orientable, as is the case for $\Sigma(\mathbb{R}P^2)$, the spherical suspension of the projective plane, whose universal cover is itself.

2. Ramified Orientable Double Covers

In this section we show that every non-orientable Alexandrov space can be obtained as the quotient of an orientable Alexandrov space by an isometric involution. Unlike in the manifold case, the involution is not required to be free. Locally non-orientable spaces arise where the involution has fixed points. We will develop the theory only for spaces without boundary, but it is easy to adapt the theory for spaces with boundary. It may help to begin by considering the simplest application of the theory.

Example 2.1. *The suspension on $\mathbb{R}P^2$ can be obtained as the quotient of S^3 by the suspension of the antipodal map on S^2.*

We will consider the matter of orientability for a more general class of topological spaces, which we refer to as *non-branching MCS spaces*. By this we mean MCS spaces where the top stratum is a connected manifold.

Our definition of orientability and local orientability for a connected, non-branching MCS space without boundary is the same as that for an Alexandrov space without boundary, allowing for Σ_p to now represent the compact connected MCS space whose cone gives us the conic neighborhood at $p \in X$.

Lemma 2.2. *Let X^n be a non-branching MCS space of dimension $n \geq 2$ without boundary. X is orientable if and only if the topological manifold $X^{(n)}$ is orientable.*

Proof. The advantage of Alexander-Spanier cohomology is that for any closed subset $A \subset X$, $H^n_c(X, A) \cong H^n_c(X \setminus A)$ Observe further that, as with Alexandrov spaces, the codimension of the singular set $S = X \setminus X^{(n)}$ of a non-branching MCS space is at least 3. Thus, as in [14], the set S is so small that

$$H^n_c(X) \cong H^n_c(X, S) \cong H^n_c(X \setminus S) \cong H^n_c(X^{(n)}).$$

Given this, one implication is trivial, and it remains to show that if $X^{(n)}$ is an orientable manifold then X is orientable. In order to do so, we must show that the generator of $H^n_c(X)$, which we identify with the fundamental class of $X^{(n)}$, induces local orientations at each point. We will do this by induction on n.

For any point p in the orientable manifold $X^{(n)}$, the generator of $H^n_c(X^{(n)})$ maps to a generator of $H^n_c(X^{(n)}, X^{(n)} \setminus \{p\})$ as usual, giving a local orientation at p. If $p \in S$, then let U be an open conic neighborhood of p, so that $U \cong K\Sigma_p X$. The set $V = U \setminus S$ is then the cone on $\Sigma_p X \setminus \Sigma_p S$. The orientation of $X^{(n)}$ induces an orientation on the $(n-1)$-dimensional manifold $\Sigma_p X \setminus \Sigma_p S$, and by the inductive hypothesis on the Alexandrov space $\Sigma_p X$. □

Theorem 2.3. *Let X be a non-branching MCS space of dimension n without boundary which is non-orientable. There there is an orientable MCS space $\tilde{X}_{Ram}$ with the same dimension and with an involution i such that $\tilde{X}_{Ram}/i \cong X$. $\tilde{X}_{Ram}$ is a ramified double cover of X, and the ramification locus is the union of those strata with non-orientable normal cones.*

Proof. Let $X^{(n)}$ be the manifold part of X. By Lemma 2.2 it is non-orientable, so it has an orientable double cover, and we can glue $X \setminus X^{(n)}$

back into the double cover to obtain a ramified double cover with the required properties. ☐

In order to apply this technique to Alexandrov spaces we will need to add metric information. We will rely on the following lemma which gives a condition for the completion of an Alexandrov domain to be an Alexandrov space. This result is a straightforward generalization of results from [17] and [36].

Lemma 2.4. *Let X be an Alexandrov domain with $\mathrm{curv} \geq k$. Then, if for every $p \in X$ a geodesic pq exists for almost all $q \in X$, the completion of X is an Alexandrov space with $\mathrm{curv} \geq k$.*

Proof. The proof is as in [17]. Let $p \in X$. For almost every geodesic γ, the geodesic $p\gamma(t)$ exists for almost every t. Let f be the function obtained from dist$(p, \cdot)$ by composition with ρ_k, where ρ_k is the function defined in Equation 1 of Section 1. It is not hard to show that f is semi-concave and in fact $f \circ \gamma$ is $(1 - kf)$-concave at almost every t (in particular, where $p\gamma(t)$ exists – see the methods of proof in [36] for further details). By 1.3 of [33], $f \circ \gamma$ is therefore $(1 - kf)$ concave for all t. By continuity, the result holds for all geodesics γ. Now f is $1 - kf$-concave on X, so that the Toponogov comparison holds globally. Once again by continuity, the comparison holds in the completion of X. ☐

The main result of this section, Theorem A, can now be seen as a corollary of Theorem 2.3 and Lemma 2.4.

Corollary 2.5. *Let X be an Alexandrov space of dimension n and $\mathrm{curv} \geq k$ without boundary which is non-orientable. There there is an orientable Alexandrov space $\tilde{X}_{Ram}$ with the same dimension and lower curvature bound, and with an isometric involution i such that $\tilde{X}_{Ram}/i$ and X are isometric. $\tilde{X}_{Ram}$ is a ramified orientable double cover of X, and the ramification locus is the union of those strata in X having non-orientable normal cones.*

Proof. The metric result is all that we need to show. We can use the involution to define a metric on $\tilde{X}_{\mathrm{Ram}}$. Within $\tilde{X}_{\mathrm{Ram}}$, the double cover of the manifold $X^{(n)}$ is an Alexandrov domain with $\mathrm{curv} \geq k$. Spaces of directions in $\tilde{X}_{\mathrm{Ram}}$ are ramified covers of spaces of directions in X, so by induction they are Alexandrov spaces with $\mathrm{curv} \geq 1$. It follows that geodesics in $\tilde{X}_{\mathrm{Ram}}$ do not branch. The codimension of the ramification locus is at least 3, and so the assumptions of Lemma 2.4 are satisfied. ☐

We will refer to $\tilde{X}_{\mathrm{Ram}}$ as the *ramified orientable double cover* of X. We now present a lemma on lifting group actions to these ramified orientable double covers, and we will also use the covers to classify positively curved spaces in dimension 3.

Lemma 2.6. *Let G be a connected Lie group acting by isometries on an n-dimensional non-orientable Alexandrov space X without boundary. Let $\tilde{X}_{Ram}$ be the ramified orientable double cover of X. Then the action of G on X lifts to an action of a 2-fold covering group of G, $\tilde{G}$ on $\tilde{X}_{Ram}$.*

Proof. Since the G-action is isometric, it must preserve the stratification of X, and in particular act on the manifold $X^{(n)}$. The action may then be lifted to an action of a 2-fold covering group on the double cover of $X^{(n)}$, which is dense in $\tilde{X}_{\text{Ram}}$. Since the action is by isometries it extends to all of $\tilde{X}_{\text{Ram}}$. □

The following classification results are an easy consequence of the theory of ramified orientable double covers. They have been obtained independently in [11], and also appear to be known to others in the field [37].

Proposition 2.7. *The only closed, simply-connected, three-dimensional Alexandrov spaces of positive curvature are homeomorphic to either S^3 or $\Sigma\mathbb{R}P^2$.*

Proof. If X is a manifold this follows from the resolution of the Poincaré Conjecture [29, 30, 31]. If X is not a manifold its ramified orientable double cover is S^3, and the only isometric involution on S^3 which fixes only isolated points yields $\Sigma\mathbb{R}P^2$ [41, 22]. □

Because there are no free isometric actions on $\Sigma\mathbb{R}P^2$, we have the following corollary.

Corollary 2.8. *Let X^3 be a closed, three-dimensional, Alexandrov space of positive curvature which is not a manifold. Then X^3 is homeomorphic to $\Sigma\mathbb{R}P^2$.*

3. Group Actions on Alexandrov Spaces

In this section, we show that some important results from the theory of isometric actions of compact Lie groups on Riemannian manifolds still hold in the context of Alexandrov geometry. We prove the Slice Theorem, and we also show that, as one would expect, the fixed point set of a group action is a totally quasigeodesic subset.

We consider torus actions on positively curved spaces, and show that their fixed point sets are always of even codimension, that in even dimensions the fixed point set is always non-empty, and that in odd dimensions if no point is fixed, then there must be a circle orbit.

3.1. Isotropy and the Slice Theorem.

Let G be a compact Lie group acting on an Alexandrov space X, and let G_p be the isotropy group at a point p. Then G_p acts in two ways: locally and infinitesimally. By acting locally,

we mean that for any $r \geq 0$, the group G_p acts on $\partial B_r(p)$. By acting infinitesimally, we mean that as $r \to 0$ this action converges to an action of G_p on the space of directions, Σ_p. Alternatively, one might also consider the natural action of G_p on the space of geodesics emanating from p, and extend it to its completion, Σ_p.

Proposition 3.1. *Let G be a compact Lie group acting by isometries on an Alexandrov space X, and let G_p be the isotropy at $p \in X$. Then for small r there are G_p-equivariant homeomorphisms $\partial B_r(p) \to \Sigma_p X$. In other words, the infinitesimal action of the isotropy group is equivalent to its local action.*

Proof. Let r be so small that the closed ball $\bar{B}_r(p) \cong \bar{K}\Sigma_p$, where $\bar{K}\Sigma_p$ denotes the closed cone on Σ_p. G_p acts on this ball. We can remove the point p and glue in Σ_p in its place to obtain a new space $W \cong \Sigma_p \times [0,1]$. Since the isotropy action is, by definition, the limit of the local action, we may combine the local and infinitesimal actions to get one continuous G_p action on W. The orbit space $\bar{W}$ of this action is obtained from $\bar{X} = X/G_p$ in a similar way, namely, we remove the point $\bar{p}$ from $\bar{B}_r(\bar{p})$ and replace it with $\Sigma_{\bar{p}}$, so that we have $\bar{W} \cong \Sigma_{\bar{p}} \times [0,1]$. Note that the orbit space has the structure of a product: by considering the isotropy action as an action on geodesics (or quasigeodesics) it is clear that the same orbit types appear as we vary $t \in [0,1]$.

We can now apply Theorem 2.5.2 of [26] (cf. Theorem II.7.1 of [4]) and we see that the action of G_p on W is equivalent to the product of the isotropy action on Σ_p and the trivial action on $[0,1]$. □

We can now prove Theorem B: the Slice Theorem for Alexandrov spaces.

Slice Theorem 3.2. *Let a compact Lie group G act isometrically on an Alexandrov space X. Then for all $p \in X$, there is some $r_0 > 0$ such that for all $r < r_0$ there is an equivariant homeomorphism $\Phi : G \times_{G_p} K\nu_p \to B_r(G(p))$, where ν_p is the space of normal directions to the orbit $G(p)$.*

Proof. Let $\pi : X \to \bar{X} = X/G$. Let $p \in X$, and let $\bar{p} = \pi(p) \in \bar{X}$. Let $\bar{h} : \bar{X} \to \mathbb{R}$ be a function built up from distance functions such that it is strictly concave on some $U \ni \bar{p}$ and attains its maximum value on U at $\bar{p}$ (see [32] for the construction). Let r_0 be such that $B_{r_0}(\bar{p}) \subset \bar{h}^{-1}([a,\infty)) \subset U$ for some a. The gradient flow of $\bar{h}$ gives the retraction $\bar{F} : B_{r_0}(\bar{p}) \to \{p\}$.

We can lift the function $\bar{h}$ to a function $h : X \to \mathbb{R}$. This function is defined by distance functions from G-orbits, and so its gradient flow gives a G-equivariant retraction $F : B_r(G(p)) \to G(p)$. Then by Proposition II.3.2 of [4], $F^{-1}(p)$ is a slice.

Since the directions of flow lines of h are horizontal with respect to π, within the space W constructed in Proposition 3.1 the slice $F^{-1}(p)$ and the

space of directions normal to the orbit, ν_p, together give a G_p-invariant subspace V, and the orbit space of the restricted action is $\Sigma_{\bar{p}} \times [0, 1]$. Therefore V is a product, and we have the result. □

Note that in the special case where X has $\mathrm{curv} \geq k > 0$ and $\partial \bar{X} \neq \emptyset$, letting $\bar{p}$ be the soul of $\bar{X}$, and $G(p)$ be the corresponding orbit in X, we may use the Sharafutdinov retraction in place of $\bar{F}$ to show that $\pi^{-1}(X \setminus \partial X)$ is equivariantly homeomorphic to $G \times_{G_p} K\nu_p$.

3.2. **Structure of Fixed Point Sets.**

Proposition 3.3. *Let G be a compact Lie group acting on an Alexandrov space X by isometries. Let $H \subset G$ be a closed subgroup, and let $F \subset X$ be the set of fixed points of H. Then F is a totally quasigeodesic subset of X, and admits a stratification into manifolds.*

Proof. The isometric image of F in the orbit space X/H is an extremal set, and therefore it is totally quasigeodesic and stratified into manifolds [32]. For any $p \in X$, the function dist$(p, \cdot)$ on F is equal to dist$(\bar{p}, \cdot)$ on the image of F, and so curves in F which are quasigeodesics for X/H are also quasigeodesics for X, giving the result. □

Example 3.4. *Suspend an isometric T^1-action on $\mathbb{C}P^2$, where T^1 fixes an S^2 and an isolated point in the $\mathbb{C}P^2$. $\mathrm{Fix}(\Sigma\mathbb{C}P^2; T^1)$ is connected and consists of an S^3 and an interval, I, where the interval's endpoints are the antipodes of S^3. The strata are (i) a twice punctured S^3, (ii) an open interval, and (iii) two isolated points.*

3.3. **Torus Actions on Positively Curved Spaces.** We first recall Petrunin's generalization of Synge's Lemma, which is used to prove the Generalized Synge's Theorem 1.7.

Generalized Synge's Lemma 3.5. [34] *Let X be an orientable Alexandrov space with $\mathrm{curv} \geq 1$ and let $T : X \to X$ be an isometry. Suppose that*

(1) *X is even-dimensional and T preserves orientation; or*
(2) *X is odd-dimensional and T reverses orientation.*

Then T has a fixed point.

In the even-dimensional case, the theory of ramified orientable double covers yields the following corollary.

Corollary 3.6. *Let X be an Alexandrov space of even dimension with $\mathrm{curv} \geq 1$, and let G be a connected Lie group acting on X by isometries. Then for any $g \in G$, g has a fixed point.*

Proof. We may take X to be non-orientable. By Lemma 2.6 we may lift the action of G to an action of a 2-fold covering group $\tilde{G}$ on $\tilde{X}_{\text{Ram}}$, and then the Generalized Synge's Lemma 3.5 applies to a lift of g which is in the connected component of the identity of $\tilde{G}$. □

Lemma 3.7. *Let T^k act by isometries on X^{2n}, an even-dimensional space of positive curvature. Then T^k has a fixed point.*

Proof. Consider a dense 1-parameter subgroup of T^k, and within it an infinite cyclic subgroup. By Corollary 3.6, the cyclic subgroup fixes a point. As we move the generator of the subgroup towards the identity, we generate a sequence of fixed points in X, and any limit point of that sequence will be fixed by the torus. □

Corollary 3.8. *Let T^k act by isometries on X^{2n+1}, an odd-dimensional space of positive curvature. Then either there is a circle orbit or T^k has a fixed point set of dimension at least one.*

Proof. If T^k has a fixed point p, then we may apply Lemma 3.7 to the isotropy action on Σ_p, a positively curved Alexandrov space of even dimension. Otherwise, let $T^1 \subset T^k$ act non-trivially, and consider the induced action of T^{k-1} on the $2n$-dimensional space X/T^1. By Lemma 3.7, this action fixes a point, and that point corresponds to a circle orbit of T^n in X. □

Finally, we note that an easy induction shows that a familiar result on the codimension of the fixed point set of circle actions (or, more generally, torus actions) on Riemannian manifolds holds for Alexandrov spaces.

Proposition 3.9. *Let T^1 act isometrically and effectively on X^n, a compact Alexandrov space. Then the fixed point set components of the circle action are of even codimension in X^n.*

4. Fixed-Point Homogeneous Actions

In positive curvature, fixed-point homogeneous Riemannian manifolds are similar to cohomogeneity one Riemannian manifolds, in that they admit a decomposition as a union of disk bundles. When one considers positively curved Alexandrov spaces of cohomogeneity one, one sees that they admit a decomposition as a union of more general cone bundles [12]. One might expect, therefore, that positively curved fixed-point homogeneous Alexandrov spaces would also admit a decomposition as a union of cone bundles, but we will see below that this is not the case.

We recall here the structure theorem for fixed-point homogeneous Riemannian manifolds in positive curvature.

Theorem 4.1 (Structure Theorem)**.** [16] *Let M be a positively curved Riemannian manifold with an (almost) effective isometric fixed-point homogeneous G-action and $M^G \neq \emptyset$. If F is the component of M^G with maximal dimension then the following hold:*

(i) *There is a unique orbit $G(p) \cong G/G_p$ at maximal distance to F (the "soul" orbit).*

(ii) *All G_p-orbits in the normal sphere S^l to $G(p)$ at p are principal and diffeomorphic to G_p/H. Moreover F is diffeomorphic to S^l/G_p.*

(iii) *There is a G-equivariant decomposition of M, as*

$$M = D(F) \cup_E D(G(p)),$$

where $D(F)$, $D(G(p))$, are the normal disk bundles to F, $G(p)$, respectively, in M with common boundary E when viewed as tubular neighborhoods.

(iv) *All orbits in $M \setminus (F \cup G(p))$ are principal and diffeomorphic to $S^k \cong G/H$, the normal sphere to F.*

Note that when X is an Alexandrov space, the normal spheres are replaced by more general spaces of positive curvature. While one would expect to decompose the space as a union of cone bundles this is not in general possible, because part (ii) of this theorem fails for Alexandrov spaces and so there may be non-principal orbits in the complement of $F \cup G(p)$.

Example 4.2. *Consider $\Sigma\mathbb{R}P^2$ with the suspended T^1 action. Here the fixed point set of the T^1 action is an interval and the soul orbit at maximal distance is a circle with $\mathbb{Z}_2$ isotropy. The $\mathbb{Z}_2$ isotropy action fixes two points, giving rise to a sphere with $\mathbb{Z}_2$ isotropy which intersects the endpoints of the fixed point set of the T^1 action at its antipodes. Although $\Sigma\mathbb{R}P^2$ is itself the union of two cone bundles over $\mathbb{R}P^2$, it is not a union of cone bundles over $\mathrm{Fix}(\Sigma\mathbb{R}P^2; T^1)$ and the soul orbit $T^1(p) \cong S^1/\mathbb{Z}_2$.*

In this section we present an alternative description of positively curved fixed-point homogeneous Alexandrov spaces as a join of a space of directions and a compact, connected Lie group, G, modulo a subgroup $K \subset G$. This provides an alternative way of viewing fixed-point homogeneous Riemannian manifolds of positive curvature.

We observe that the maximal connected component of the fixed point set of a fixed-point homogeneous action has codimension 1 in the orbit space and corresponds to a union of faces in the boundary. It follows that it is unique in positive curvature [34], provided $\dim(X/G) > 1$.

For Alexandrov spaces that admit an isometric, fixed-point homogeneous G-action for which $X^G \neq \emptyset$, we can now prove Theorem C as stated in the introduction.

Proof of Theorem C. Part (i) follows from the Soul Theorem 1.6 applied to the quotient space X/G, a positively curved Alexandrov space, retracting from the faces of the boundary which make up F. Let ν be the space of normal directions to the orbit $G(p)$. Part (ii) follows from the Slice Theorem 3.2, noting also that F is homeomorphic to the space of directions at the soul point of X/G, which is ν/G_p.

For part (iii), $X \setminus F$ is homeomorphic to $(K(\nu) \times G)/G_p$ by the Slice Theorem, and this homeomorphism is G-equivariant, where G acts trivially on ν and acts on G by its left action. We may write $K(\nu) \times G$ as $\nu \times G \times (0, 1]$, where $\nu \times G \times \{1\}$ is identified to G. The set F is, by part (ii), homeomorphic to ν/G_p, and it is fixed by G, so the entire space X is in fact homeomorphic to $(\nu \times G \times [0, 1])/G_p$, where $\nu \times G \times \{0\}$ has been identified to ν and $\nu \times G \times \{1\}$ to G, and the homeomorphism is G-equivariant.

For part (iv), let $\bar{p} \in F \subset X/G$ be F-regular and the image in ν/G_p of a principal G_p-orbit. Then at $p \in F \subset X$ we have an isometry $\Sigma_p X = S^{k-1} * N$ where $k = \dim(F)$ and N is the normal space. There is a neighborhood of p which is comprised entirely of principal orbits and of fixed points in F. Since by assumption G acts transitively on N, N is given by G/H. It follows that G/H has curvature bounded below by 1, completing the proof of the theorem. □

In the special case where $H \lhd G_p \subset G$, we have the following corollary.

Corollary 4.3. *Let G act isometrically and fixed-point homogeneously on X^n, an n-dimensional, closed Alexandrov space of positive curvature and assume that $X^G \neq \emptyset$. Suppose further that $H \lhd G_p \subset G$, where H is the principal isotropy of the G-action and G_p is the isotropy subgroup of the "soul" orbit. Then X^n is equivariantly homeomorphic to*

$$(\nu * G/H)/K,$$

where ν is the normal space of directions to G/G_p and $K \cong G_p/H$. That is, X^n is homeomorphic to the quotient of the join of two positively curved Alexandrov spaces.

Proof. Write K for G_p/H. By Theorem C, X^n is homeomorphic to

$$(\nu * G)/G_p \cong ((\nu * G)/H)/K.$$

Since H is the principal isotropy of the G_p-action on ν, and is a normal subgroup of G_p, it acts trivially on ν. Therefore we have

$$(\nu * G)/H \cong (\nu * G/H),$$

and the result follows. □

Note that for any space ν of $\mathrm{curv} \geq 1$, we can join ν to any homogeneous G-space of $\mathrm{curv} \geq 1$ to obtain a positively curved Alexandrov space

with a fixed-point homogeneous G-action. In this sense, we can think of fixed-point homogeneous spaces as being plentiful among positively curved Alexandrov spaces.

Observe, however, that if we restrict our attention to positively curved Riemannian n-dimensional manifolds and assume that $G_p \neq G$, then, with the unique exception of the fixed-point homogeneous $Spin(9)$-action on CaP^2, $H \lhd G_p \subset G$. Hence, Corollary 4.3 allows us to represent all such manifolds as

$$M^n \cong (S^k * G/H)/G' \cong (S^k * S^l)/G' \cong S^{k+l+1}/G',$$

where $G' \cong G_p/H$, and G' is one of either $SU(2)$, $N_{SU(2)}(T^1)$, T^1, or a finite subgroup of $O(n+1)$ (cf. [4]), that is, all these manifolds are spheres or the base of a fibration whose total space is a sphere. CaP^2 cannot be written as the base of such a fibration, and its decomposition as a join is given by

$$(S^7 * Spin(9))/Spin(8).$$

We also note that in the special case where $G_p = G$, the principal isotropy subgroup is almost always *not* normal in G. This is not an issue for the Riemannian case, though, because the only groups that can act principally on n-spheres must either act transitively, in which case, the decomposition as a join gives us that

$$(\nu * G)/G_p \cong S^0 * G/H \cong S^0 * S^l = S^{l+1},$$

or they must act freely, as in the examples discussed above.

Finally, we note that in analogy to the Riemannian case, we can decompose a fixed-point homogeneous, positively curved Alexandrov space as a union of cone bundles when we assume that all orbits in the complement of the fixed point set F and the soul orbit $G(p)$ are principal. That is, we have the following corollary whose proof we leave to the reader.

Corollary 4.4. *Let G act isometrically and fixed-point homogeneously on X^n, an n-dimensional, closed, Alexandrov space of positive curvature and assume that $X^G \neq \emptyset$. Let $H \subset G$ be the principal isotropy and let F be the component of X^G with maximal dimension. If we assume further that the orbits in $X \setminus (F \cup G(p))$ are principal, then X^n decomposes as*

$$K_{G/H}(F) \cup K_\nu(G(p)),$$

that is, as the union of cone bundles over F and $G(p)$ having as fibers cones on G/H, a positively curved homogeneous space, and ν, the positively curved normal space of directions to $G(p)$, respectively. In particular, in the Riemannian case, the fibers are cones on spheres, and so we have disk bundles.

5. Maximal Symmetry Rank

In this section we show that, as in the Riemannian case, the bound on the symmetry rank of a positively curved Alexandrov space of dimension n is $\lfloor \frac{n+1}{2} \rfloor$, and that when the bound is achieved some circle subgroup acts fixed-point homogeneously. We then inductively use the join description of fixed-point homogeneous spaces to show that all such spaces are quotients of spheres as stated in Theorem D.

Theorem 5.1. *Let T^k act isometrically and (almost) effectively on X^n, a positively curved Alexandrov space. Then,*

$$k \leq \lfloor \frac{n+1}{2} \rfloor.$$

Further, in the case of equality, for some $T^1 \subset T^k$, $\mathrm{codim}(\mathrm{Fix}(X^n; T^1)) =$ 2.

Proof. If X has boundary then the action is determined by the isotropy at the soul, and if X is non-orientable the action will lift to the ramified orientable double cover. Therefore we may assume that X is closed and orientable.

The proof is by induction on the dimension n of the space. Where $n = 1$, the maximal torus action is the free action of T^1 on S^1, fixing the empty set, $\emptyset$, which has codimension 2. The crux of the induction step is that where a group acts effectively, the action of any isotropy group on a normal space must also be effective [10].

If $n = 2k-1$, then by induction an effective action of T^k (or of a torus of higher rank) cannot fix points, and so Corollary 3.8 implies that the action has a circle orbit. If $n = 2k$, then Lemma 3.7 implies that the action has a fixed point.

Aiming for a contradiction, we suppose that T^{k+1} acts on X^n, with $n = 2k-1$ or $2k$. Consider the isotropy action at a circle orbit or at a fixed point, respectively. By the inductive hypothesis, this action cannot be effective. This proves the bound on the rank. If T^k acts, we again consider the isotropy at a circle orbit or fixed point. This action is also of maximal rank and so there is a circle subgroup of the isotropy which fixes a set of codimension 2 in the normal space. This subgroup will also fix a set of codimension 2 in X. $\square$

Example 5.2. *The standard unit sphere S^n has isometry group $O(n+1)$, and this group is clearly of maximal rank. We may consider the action of the maximal torus in $O(n+1)$ to be the prototypical maximal rank action in positive curvature.*

Example 5.3. *Let Γ be a finite subgroup of the maximal torus in $O(n+1)$. Then clearly S^n/Γ admits an action of maximal symmetry rank. If we require Γ to act freely we obtain the lens spaces (including the odd dimensional projective spaces)* [42]*. Note that in this category of examples, the $2n$-dimensional spaces are simply suspensions of the $(2n-1)$-dimensional spaces.*

Example 5.4. *It is not necessary that Γ be a subgroup of the maximal torus: it is enough that it commute with the torus. This creates a distinction only in $O(2n+1)$, where the maximal torus T^n commutes with the antipodal map. Therefore we may pick $\Gamma \subset Z(T^n) \subset O(2n+1)$. If we require Γ to act freely, we obtain the even-dimensional projective spaces. However, if Γ fixes points we will obtain spaces which are locally non-orientable, such as $S^4/\mathbb{Z}_2 \cong \mathbb{R}P^2 * S^1$, where the involution fixes a circle. We can also obtain spaces with boundary. For example, if Γ is simply a reflection then the quotient space is a disk.*

Example 5.5. *Let $\Gamma \subset T^{n+1} \subset O(2n+2)$ be rank one. Then the $2n$-dimensional space S^{2n+1}/Γ admits a T^n-action. In particular, if Γ acts freely (and so is the diagonal circle) then we obtain a T^n-action on $\mathbb{C}P^n$.*

We see that many spaces of maximal symmetry rank can be obtained in the same way as the Riemannian examples, that is, by taking quotients of spheres. We obtain non-Riemannian spaces simply by allowing the group to have isotropy. Theorem D shows that all such spaces arise in this way.

Theorem D. *Let X be an n-dimensional, compact, Alexandrov space with $\operatorname{curv} \geq 1$ admitting an isometric, (almost) effective T^k-action. Then $k \leq \lfloor \frac{n+1}{2} \rfloor$ and in the case of equality either*

(1) *X is a spherical orbifold, homeomorphic to S^n/G, where G is a finite subgroup of the centralizer of the maximal torus in $O(n+1)$ or*
(2) *only in the case that n is even, $X \cong S^{n+1}/G$, where G is a rank one subgroup of the maximal torus in $O(n+2)$.*

In both cases the action on X is equivalent to the action induced by the maximal torus on the G-quotient of the corresponding sphere.

Proof. The bound on the rank has already been shown. Let us assume that X^n is closed and orientable, and that T^k acts with maximal rank on X. By Theorem 5.1, there is some $T^1 \subset T^k$ which acts with a fixed point set of codimension 2. Let F be the fixed point set. Then there is a unique orbit $T^1(p)$ at maximal distance from F. Let ν_p be the normal space to this orbit. In X/T^1, this orbit becomes a point fixed by T^k/T^1. We either have $T^1_p = T^1$, in which case p is fixed by the entire torus, or T^1_p is finite,

in which case $T^k(p)$ is a circle orbit. In either case, the isotropy action of T^k_p on ν_p is once again an action of maximal symmetry rank upon a closed orientable positively curved Alexandrov space.

Now by Theorem C, we may write $X = (\nu * S^1)/T^1_p$. We proceed inductively, until ν is given by S^1 or S^0, to see that X is homeomorphic to the quotient of some sphere S^m by a subgroup G of a linearly acting torus, and that the torus action on X is induced by the action of the torus on S^m. It is clear from considering the dimension of the space and the rank of the torus that either $m = n$ and $\mathrm{rk}(G) = 0$, or that, only in the case where n is even, $m = n + 1$ and $\mathrm{rk}(G) = 1$. Further, one easily sees that the orbit space X/T^k, stratified by isotropy type, is either a simplex or a suspended simplex.

In the case that X^n is not orientable, we have $X = (S^m/G)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ reverses orientation. It follows that $(S^m/G)/T^k$ is a suspended simplex, and $\dim(X) = m = n$ is even with G finite. The action of $\mathbb{Z}_2$ can then be lifted to the sphere, where it is the composition of an element of the torus with a reflection. In the case that X has boundary, the isotropy action at the soul determines the action on X. X is the cone on an odd-dimensional maximal symmetry rank space, or, equivalently, the quotient by a reflection of the suspension of an odd-dimensional maximal symmetry rank space. $\square$

We can see from the inductive nature of the proof that if the space is odd-dimensional then it is built up by an iterated process, joining a circle to a space of lower dimension and then taking a quotient by a finite group, that is one can write it as

$$(\cdots(S^1/\Gamma_1 * S^1)/\Gamma_2 * \cdots * S^1)/\Gamma_k,$$

where Γ_i are finite subgroups of T^k, $1 \leq i \leq k$. If it is even-dimensional and orientable, then it is a suspension of an odd-dimensional example, or it is the quotient of an odd-dimensional example by a circle. If it is not orientable, or has boundary, then it is a quotient by an involution on an example of suspension type which interchanges the poles of the suspension.

This view of the proof allows us to show the following result on the fundamental groups of these spaces.

Proposition 5.6. *Let X^n be a positively curved Alexandrov space of maximal symmetry rank. Then if n is odd X has finite cyclic fundamental group and $\chi(X) = 0$. If n is even then X is contractible if it has boundary, is simply connected if it is orientable or locally non-orientable, and has fundamental group of order two if it is locally orientable but not globally orientable. The Euler characteristic $\chi(X) = 2$ or $n + 1$ if X is closed and orientable and $\chi(X) = 1$ otherwise.*

Proof. The odd-dimensional spaces are spherical orbifolds, and from the proof of Theorem 5 we can see that the first isotropy group T^1_p, which is at most finite cyclic, is the only group which might act without fixing points, so by [1] the result follows. The other cases are either trivial, or consequences of the Generalized Synge's Theorem 1.7.

Because X is homeomorphic to the orbit space of an isometric action on a Riemannian manifold, the fact that $\chi(X) = \chi(\mathrm{Fix}(X; T^1))$ for any T^1 in the torus now follows just as in the Riemannian case [21] (noting that the subspaces to which the Lefschetz fixed point theorem are applied are all triangulable by [19]). The result is then true by induction. □

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